Section 3.4: Bases

- A subset S of a vector space V is called a **basis** of V if S is a linearly independent, spanning set for V.
- Many familiar vector spaces like $\mathbb{R}^m$, $\mathcal{P}$, and $M_{m,n}(\mathbb{R})$ have *standard bases*, but sometimes it will be useful to consider other bases, as well. We will also consider the problem of finding a basis for a given subspace of one of these spaces.

Bases in $\mathbb{R}^m$

Theorem 3.34:

1. Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a subset of $\mathbb{R}^m$ and let A be the $m \times n$ matrix whose columns are the vectors in S. The set S is basis of $\mathbb{R}^m$ if and only if and $\operatorname{rref}(A) = I_n$, (in which case $n = m$).
2. Every basis of $\mathbb{R}^m$ has m elements.
3. Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be a subset of $\mathbb{R}^m$ and let A be the $m \times m$ matrix whose columns are the vectors in S. The set S is basis of $\mathbb{R}^m$ if and only if $\det(A) \neq 0$.

Example: Is $\{\langle 1, 3, 2\rangle, \langle 2, 1, 0\rangle, \langle 1, -1, 0\rangle\}$ a basis of $\mathbb{R}^3$?

Properties of Bases

Theorem 3.38: Let S be a subset of a vector space V.

1. If S is linearly independent, but every set T with $S \subset T \subseteq V$ is linearly dependent, then S is a basis of V. In other words, *a maximal linearly independent subset of V is a basis of V*.
2. If S spans V, but every subset $T \subset S$ does not span V, then S is a basis of V. In other words, *a minimal spanning set of V is a basis of V*.
3. S is a basis of V if and only if every vector in V can be written uniquely as a linear combination of vectors in S.

Properties of Bases

Theorem 3.39: Let V be a vector space, and assume that V has a spanning set S with m elements. Let $T \subseteq V$ be a (finite or infinite) set with n elements where $n > m$. Then T is linearly dependent.

Theorem 3.40: Let V be a vector space. Then every basis of V has the same number of elements.

Dimension

- Let V be a vector space. If V has a basis with a finite number n elements, we say that n is the **dimension** of V. In this case, we say V is **finite dimensional**, and we write $\dim V = n$. If V does not have a finite basis, we say that V is **infinite dimensional**, and we write $\dim V = \infty$.
- The dimensions of some vector spaces:

  $\dim \mathbb{R}^m = m$

  $\dim \mathcal{P} = \infty$

  $\dim \mathcal{P}_n = n + 1$

  $\dim M_{m,n}(\mathbb{R}) = mn$